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CONSTRUCTION OF PIECEWISE-ANALYTICAL GAS FLOWS JOINED THROUGH SHOCK WAVES NEAR THE AXIS OR CENTER OF SYMMETRY

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Analytical solutions of a quasilinear system of equations with partial derivatives are constructed in the case where the initial data for different functions are specified on different surfaces and the resultant problem has singularities of the form u/x and w/x. Conditions for existence and uniqueness of a solution in the form of formal power series for the problem posed and sufficient conditions for convergence of the series are indicated. A generalization of the problem considered is given. Results of the study are used to solve the problem of the focussing of a compression wave generated by a piston moving smoothly in a quiescent gas: a solution for $t = 0$, including determination of the piston trajectory, and a solution for $t < 0$, including unequivocal construction of the front of a reflected shock wave, are uniquely constructed from the distribution of gas-dynamic quantities for $t > 0$. The solution of this problem is a generalization to the case of two independent variable self-similar Sedov's solutions.

Introduction. In a class of self-similar flows that depend on one independent variable $\lambda = r/t$, solutions are known that describe the focusing of a compression wave generated by a piston moving smoothly by a specified law in a homogeneous quiescent gas [1–3]. After focussing of a weak discontinuity, a reflected shock wave (SW), behind which the compressed gas is at rest, propagates at a finite constant speed from the center or axis of symmetry. Using the characteristic Cauchy problem [4], for the non-self-similar problem of a piston moving smoothly in a gas from a point $r = r_0$, Bautin [5] obtained a local solution in a vicinity of the point ($t = t_0$, $r = r_0$) for $r_0 > 0$ [5]. In a class of self-similar flows dependent on one variable $\xi = t/r^k$ ($1 < k < 2$), Godunov and Kireeva [6] and Zababakhin and Zababakhin [7] constructed flows with shock waves reflected from the point $r = 0$ [6, 7], whose speed is variable and is equal to infinity at the moment $t = 0$. Teshukov [8, 9] proved the existence and uniqueness of piecewise-analytical solutions for the problem of the reflection of a spatial SW from a curvilinear wall and for the problem of the interaction of curvilinear SW fronts. For cylindrically and spherically symmetric flows, Bautin and Kazakov [10] constructed a solution of the problem of the reflection of a SW from the axis or center of symmetry. In this case, the SW moves at a finite speed, but the question of joining of the flows behind and ahead of the SW remains to be solved.

In studies of gas flows with reflected shock waves, it is necessary to solve generalized Cauchy problems [11–13], including those for systems with a singularity [14].

1. Generalized Cauchy Problem. We consider a generalized Cauchy problem of the form

$$w_x = \left[r(x,y,u,v,w)\,\frac{u}{x} + s(x,y,u,v,w)\,\frac{w}{x} + t(x,y,u,v,w)\right]\Big|_{y=0}, \tag{1.1}$$

$$u_x = a(x,y,u,v,w)u_y + b(x,y,u,v,w)v_x + \frac{u}{x}f(x,y,u,v,w) + e(x,y,u,v,w)w_x + p(x,y,u,v,w),$$

$$v_y = c(x,y,u,v,w)u_y + d(x,y,u,v,w)v_x + \frac{u}{x}g(x,y,u,v,w) + h(x,y,u,v,w)w_x + q(x,y,u,v,w),$$

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$$w(x)\Big|_{x=0} = 0, \quad u(x,y)\Big|_{x=0} = 0, \quad v(x,y)\Big|_{y=0} = 0.$$

Here u, v, and w are unknown functions and x and y are independent variables. For brevity, the generalized Cauchy problem is called problem A below.

Theorem 1.1. *Let, in problem* (1.1), *the functions a, b, c, d, e, f, g, h, p, q, r, s, and t be analytical in a vicinity of the point O ($x = 0$, $y = 0$, $u = 0$, $v = 0$, $w = 0$). We introduce the constants*

$$A_0 = a(O), \quad B_0 = b(O), \quad C_0 = c(O), \quad D_0 = d(O),$$

$$f_0 = f(O), \quad g_0 = g(O), \quad E_0 = e(O), \quad H_0 = h(O), \quad r_0 = r(O), \quad s_0 = s(O),$$

$$A_n = \frac{nA_0}{n - f_0}, \quad B_n = \frac{nB_0}{n - f_0}, \quad C_n = C_0 + \frac{A_0 g_0}{n - f_0}, \quad D_n = D_0 + \frac{B_0 g_0}{n - f_0}, \tag{1.2}$$

$$C_n^* = C_0 + \left(g_0 + \frac{H_0 r_0 n}{n - s_0}\right)\frac{A_0}{(n - f_0 - r_0 E_0 n/(n - s_0))}.$$

The numerical sequences δ_n, Δ_n, and Δ_n^* are given by the formulas

$$\delta_0 = 0, \quad \Delta_0 = 1, \quad \Delta_0^* = 1, \quad \delta_{n+1} = 1 + A_{n+1}D_{n+1}B_n\delta_n/(B_{n+1}\Delta_n),$$

$$\Delta_{n+1} = 1 - C_{n+2}B_{n+1}\delta_{n+1}, \quad \Delta_{n+1}^* = 1 - C_{n+2}^*B_{n+1}\delta_{n+1} \quad \text{for } B_0 \neq 0,\ n \in N, \tag{1.3}$$

$$\Delta_n = 1, \quad \Delta_n^* = 1, \quad \delta_n = 1 \quad \text{for } B_0 = 0,\ n \in N.$$

If the conditions

$$f_0 \neq n, \quad s_0 \neq n, \quad f_0 + \frac{r_0 E_0 k}{k - s_0} \neq 0, \quad \Delta_n^* \neq 0, \quad \Delta_n \neq 0, \quad n, k \in N; \tag{1.4}$$

$$\lim_{n\to\infty} \delta_n = \delta_\infty, \quad |\delta_\infty| < +\infty, \quad \lim_{n\to\infty} \Delta_n = \Delta_\infty \neq 0, \quad |\Delta_\infty| < +\infty; \tag{1.5}$$

$$|A_0 D_0|/\Delta_\infty^2 < 1, \tag{1.6}$$

are satisfied, problem (1.1) *has a unique locally analytical solution.*

Remark. If conditions (1.4) and (1.5) are satisfied, the equalities $\lim_{n\to\infty} C_n = \lim_{n\to\infty} C_n^* = C_0$ and $\lim_{n\to\infty} \Delta_n^* = \lim_{n\to\infty} \Delta_n = \Delta_\infty$ are valid.

Proof. Before constructing a solution of problem (1.1) in the form of formal power series and then proving their convergence, in problem (1.1) we change the variables: $x' = \varepsilon_1 x$, $y' = \varepsilon_2 y$, and $\varepsilon_1, \varepsilon_2 = \text{const} > 0$. In the new problem A obtained as a result of this change, values of the constants Δ_n', $(\Delta_n^*)'$, δ_n', Δ_∞', and δ_∞' coincide with values of the corresponding constants (1.2) of problem (1.1). However, because of the special choice of ε_1 and ε_2, the following inequalities hold:

$$|a'(O)| = |\varepsilon_2 a(O)/\varepsilon_1| < |\Delta_\infty|, \qquad |d'(O)| = |\varepsilon_1 d(O)/\varepsilon_2| < |\Delta_\infty|.$$

In what follows, it is assumed that the corresponding change has been made. In the new problem, the primes are omitted to simplify the notation, i.e., the notation of (1.1) is used for the new problem. Then, the following inequalities hold:

$$|A_0| < |\Delta_\infty|, \qquad |D_0| < |\Delta_\infty|. \tag{1.7}$$

We construct a solution of problem (1.1) in the form of series (the quantity z takes values u and v)

$$z(x,y) = \sum_{k,l\in N_0} z_{k,l}x^k y^l/(k!l!), \quad z_{k,l} = \frac{\partial^{k+l}z}{\partial x^k \partial y^l}\Big|_{x=y=0}, \tag{1.8}$$

$$w(x) = \sum_{n\in N_0} w_n x^n/n!, \qquad w_k = \frac{d^k w}{dx^k}\Big|_{x=0}.$$

We introduce the following designations:

$$t^* = [(r - r_0)u + (s - s_0)w + xt]\Big|_{y=0},$$

$$p^* = [(a-A_0)u_y + (b-B_0)v_x]x + (f-f_0)u + e(ru+sw+xt)\Big|_{y=0} - E_0(r_0u+s_0w)\Big|_{y=0} + xp,$$

$$q^* = [(c-C_0)u_y + (d-D_0)v_x]x + (g-g_0)u + h(ru+sw+xt)\Big|_{y=0} - H_0(r_0u+s_0w)\Big|_{y=0} + xq.$$

We multiply both sides of the equations of system (1.1) by x. In the designations introduced above, the system takes the form

$$\begin{gathered} xw_x = (r_0u + s_0w + t^*)\Big|_{y=0}, \\ xu_x = x(A_0u_y + B_0v_x) + uf_0 + E_0(r_0u+s_0w)\Big|_{y=0} + p^*, \\ xv_y = x(C_0u_y + D_0v_x) + ug_0 + H_0(r_0u+s_0w)\Big|_{y=0} + q^*. \end{gathered} \tag{1.9}$$

The superscript asterisk at t^*, p^*, and q^* in problem (1.9) is omitted for convenience. We study the problem

$$\begin{gathered} xw_x = (r_0u+s_0w+t)\Big|_{y=0}, \quad xu_x = x(A_0u_y+B_0v_x) + uf_0 + E_0(r_0u+s_0w)\Big|_{y=0} + p, \\ xv_y = x(C_0u_y+D_0v_x) + ug_0 + H_0(r_0u+s_0w)\Big|_{y=0} + q, \quad w(0)=0, \quad u(0,y)=0, \quad v(x,0)=0. \end{gathered} \tag{1.10}$$

We set

$$r_{k,l} = \frac{\partial^{k+l} r}{\partial x^k\,\partial y^l}\bigg|_{\substack{x=y=0\\ u=v=w=0}}, \quad t_n = \frac{\partial^n t\big|_{y=0}}{\partial x^n}\bigg|_{x=u=v=w=0},$$

$$\mathbf{z}_n = (z_{n,0}, z_{n-1,1}, \ldots, z_{0,n}), \quad \mathbf{r}_n = (r_{n,0}, r_{n-1,1}, \ldots, r_{0,n}).$$

The quantity r takes values p and q. Components of the vectors $(\mathbf{z}_m, w_m)$ for $0 \leqslant m \leqslant l+k=n$ enter in $r_{k+1,l}$, t_{n+1}, and components of the vectors $(\mathbf{z}_m, w_m)$ for $m > l+k=n$ do not enter in them.

The possibility of unique determination of the coefficients of series (1.8) is proved by induction with respect to $n = k+l$. By virtue of the initial conditions,

$$u_{0,l} = v_{k,0} = w_0 = 0 \text{ for all } k,l \in N_0 \tag{1.11}$$

and, in particular, $w_0 = u_{0,0} = v_{0,0} = 0$. Hence, $\mathbf{z}_0$ and w_0 are uniquely determined by the initial conditions. Let all $\mathbf{z}_0, w_0, \ldots, \mathbf{z}_n, w_n$ $(n \geqslant 0)$ be found. To calculate $\mathbf{z}_{n+1}$ and w_{n+1}, we differentiate Eqs. (1.10) $k+1$ times with respect to x and $n-k$ times with respect to y (setting $x=y=u=v=w=0$) and take into account the initial conditions (1.11) and the fact that w does not depend on y. As a result, we obtain the relations

$$\begin{gathered} (n+1)w_{n+1} = r_0u_{n+1,0} + s_0w_{n+1} + t_n, \qquad u_{0,n+1} = 0, \\ u_{1,n} = A_0u_{0,n+1} + B_0v_{1,n} + f_0u_{1,n} + p_{1,n}, \qquad v_{0,n+1} = C_0u_{0,n+1} + D_0v_{1,n} + g_0u_{1,n} + q_{1,n}, \\ \ldots \\ (k+1)u_{k+1,n-k} = (k+1)A_0u_{k,n-k+1} + (k+1)B_0v_{k+1,n-k} + f_0u_{k+1,n-k} + p_{k+1,n-k}, \\ (k+1)v_{k,n+1-k} = (k+1)C_0u_{k,n+1-k} + (k+1)D_0v_{k+1,n-k} + g_0u_{k+1,n-k} + q_{k+1,n-k}, \\ \ldots \\ nu_{n,1} = nA_0u_{n-1,2} + nB_0v_{n,1} + f_0u_{n,1} + p_{n,1}, \\ nv_{n-1,2} = nC_0u_{n-1,2} + nD_0v_{n,1} + g_0u_{n,1} + q_{n,1}, \\ (n+1)u_{n+1,0} = (n+1)A_0u_{n,1} + (n+1)B_0v_{n+1,0} + f_0u_{n+1,0} + E_0r_0u_{n+1,0} + E_0s_0w_{n+1} + p_{n+1,0}, \\ (n+1)v_{n,1} = (n+1)C_0u_{n,1} + (n+1)D_0v_{n+1,0} + g_0u_{n+1,0} + H_0r_0u_{n+1,0} + H_0s_0w_{n+1} + q_{n+1,0}, \\ v_{n+1,0} = 0, \end{gathered} \tag{1.12}$$

which are a system of linear algebraic equations for $\mathbf{u}_{n+1}$, $\mathbf{v}_{n+1}$, and w_{n+1}.

System (1.12) is solved by the method of successive elimination of unknowns.

The "direct process" consists of constructing the quantities

$$\Delta_0 = 1, \quad \delta_1 = 1, \quad \Delta_1 = 1 - C_2 B_1 \delta_1, \quad \Delta_1^* = 1 - C_2^* B_1 \delta_1,$$

$$\delta_{k+1} = 1 + \frac{B_k}{B_{k+1}} A_{k+1} D_{k+1} \frac{\delta_k}{\Delta_k} \text{ for } B_0 \neq 0 \text{ or } \delta_{k+1} = 1 \text{ for } B_0 = 0,$$

$$\Delta_{k+1} = 1 - C_{k+2} B_{k+1} \delta_{k+1}, \quad \Delta_{k+1}^* = 1 - C_{k+2}^* B_{k+1} \delta_{k+1}, \quad k = 2, \dots, n,$$

$$\psi_{1,n} = Q_{0,n}/\Delta_0, \quad \chi_{1,n} = P_{0,n}, \quad \psi_{2,n-1} = (C_2 \chi_{1,n} + Q_{1,n-1})/\Delta_1,$$

$$\chi_{k+1,n-k} = A_{k+1}(B_k \delta_k \psi_{k+1,n-k} + \chi_{k,n-k+1}) + P_{k,n-k},$$

$$\psi_{k+1,n-k} = (C_{k+1} \chi_{k,n+1-k} + Q_{k,n-k})/\Delta_k, \quad k = 1, \dots, n,$$

$$\psi_{n+1}^* = (C_{n+1}^* \chi_{n,1} + Q_n^*)/\Delta_n^*, \quad \chi_{n+1}^* = A_{n+1}^*(B_n \delta_n \psi_{n+1}^* + \chi_{n,1}) + P_n^*.$$

Here

$$P_{k,n-k} = \frac{p_{k+1,n-k}}{k+1-f_0}; \quad Q_{k,n-k} = \frac{q_{k+1,n-k}}{k+1} + \frac{g_0 p_{k+1,n-k}}{(k+1)(k+1-f_0)},$$

$$P_n^* = \frac{p_{n+1,0} + E_0 s_0 t_{n+1}/(n+1-s_0)}{n+1-f_0-r_0 E_0 (n+1)/(n+1-s_0)},$$

$$Q_n^* = \frac{1}{n+1}\left[q_{n+1,0} + \frac{H_0 s_0 t_{n+1}}{n+1-s_0}\right] + \frac{1}{n+1}\left[g_0 + \frac{H_0 r_0 (n+1)}{n+1-s_0}\right] \frac{p_{n+1,0} + E_0 s_0 t_{n+1}/(n+1-s_0)}{n+1-f_0-r_0 E_0(n+1)/(n+1-s_0)}.$$

The "reverse process" is $v_{n+1,0} = 0$, $v_{n,1} = \psi_{n+1}^*$, $u_{n+1,0} = \chi_{n+1}^*$,

$$v_{n-1,2} = \frac{D_n}{\Delta_{n-1}} v_{n,1} + \psi_{n,1}, \quad u_{n,1} = B_n \delta_n v_{n,1} + \chi_{n,1},$$

$$\dots$$

$$v_{0,n+1} = \frac{D_1}{\Delta_0} v_{1,n} + \psi_{1,n}, \quad u_{1,n} = B_1 \delta_1 v_{1,n} + \chi_{1,n}.$$

Before proving the convergence of series (1.8), we transform the recursive formulas for $\chi_{k+1,n-k}$ to explicit expressions in terms of A_i, B_i, Δ_i, Δ_n^*, δ_i, $P_{k,l}$, and $Q_{k,l}$:

$$\chi_{1,n} = P_{0,n}, \psi_{2,n-1} = (C_2 P_{0,n} + Q_{1,n-1})/\Delta_1,$$

$$\chi_{k+1,n-k} = \left\{\sum_{i=1}^{k}\left[\left(\prod_{j=i}^{k} \frac{A_{j+1}}{\Delta_j}\right) P_{i-1,n+1-i}\right] + P_{k,n-k}\right\} + \left\{\sum_{i=1}^{k}\left[\left(\prod_{j=i}^{k} \frac{A_{j+1}}{\Delta_j}\right) B_i \delta_i Q_{i,n-i}\right]\right\}, \quad k = 0, \dots, n-1,$$

$$\chi_{n+1}^* = \left\{\sum_{i=1}^{n}\left[\frac{A_{n+1}^*}{\Delta_n^*}\left(\prod_{j=i}^{n-1} \frac{A_{j+1}}{\Delta_j}\right) P_{i-1,n+1-i}\right] + P_n^*\right\} + \left\{\sum_{i=1}^{n-1}\left[\frac{A_{n+1}^*}{\Delta_n^*}\left(\prod_{j=i}^{n-1} \frac{A_{j+1}}{\Delta_j}\right) B_i \delta_i Q_{i,n-i}\right]\right\} + \frac{A_{n+1}^*}{\Delta_n^*} B_n \delta_n Q_n^*.$$

We now express $v_{k,n-k+1}$ in terms of $\psi_{i,n-i}$:

$$v_{n,1} = \psi_{n+1}^*, \qquad v_{n-1,2} = \frac{D_n}{\Delta_{n-1}} \psi_{n+1}^* + \psi_{n,1},$$

$$\dots$$

$$v_{k,n-k+1} = \sum_{i=k+1}^{n-1}\left[\left(\prod_{j=k+1}^{i} \frac{D_j}{\Delta_{j-1}}\right) \psi_{i+1,n-i}\right] + \psi_{k+1,n-k} + \left(\prod_{j=k+1}^{n} \frac{D_j}{\Delta_{j-1}}\right) \psi_{n+1}^*, \quad k = 0, 1, \dots, n.$$

The convergence of series (1.8) is proved by the method of majorants.

From conditions (1.5) and (1.7) it follows that there are constants M_1 and q_* such that for all $k, l \in N_0$ we have

$$M_1 \geqslant 1, \qquad 0 < q_* < 1, \qquad \prod_{i=n}^{k+n} \frac{|A_{i+1}|}{|\Delta_i|} \leqslant M_1 q_*^k, \qquad \frac{|A_{n+1}^*|}{|\Delta_n^*|} \leqslant M_1,$$

$$\prod_{i=n}^{n+k} \frac{|D_{i+1}|}{|\Delta_i|} \leqslant M_1 q_*^k, \qquad |r_n| < M_1, \qquad \frac{1}{|\Delta_k|} \leqslant M_1, \qquad \frac{1}{|\Delta_k^*|} \leqslant M_1, \qquad |B_n \delta_n| \leqslant M_1,$$

$$\frac{|C_{n+1}|}{|\Delta_n|} \leqslant M_1, \qquad |A_k| \leqslant M_1, \qquad \frac{|C_{n+1}^*|}{|\Delta_n^*|} \leqslant M_1, \qquad |A_k^*| \leqslant M_1..$$

As a consequence, with an appropriate choice of M_2 and M_3, and for $\rho > 0$, the Cauchy problem

$$\begin{aligned} Z_\tau^* &= \frac{M_2}{[1 - (\tau + 2Z^* + W^*)/\rho]} [(\tau + 2Z^* + W^*)(2Z_\tau^* + W_\tau^*) + 1], \quad Z^*(0) = 0, \\ W_\tau^* &= \frac{M_3}{[1 - (\tau + 2Z^* + W^*)/\rho]} [(\tau + 2Z^* + W^*)(2Z_\tau^* + W_\tau^*) + 1], \quad W^*(0) = 0 \end{aligned} \tag{1.13}$$

majors the solution of problem (1.1). Here U^* majors u, v ($U^* \gg u, v$), $W^* \gg w$, $W^* \gg z$, and $\tau = x + y$.

Writing the differential system of problem (1.13) in normal form, we find that the right sides of this system are analytical functions that major zero, and, hence, for problem (1.13), the Cauchy–Kowalewski theorem is valid. Therefore, problem (1.13) has an analytical solution that majors series (1.8). The proof of Theorem 1.1 is complete.

Let us formulate and prove the sufficiency of some conditions for the validity of Theorem 1.1, and generalizations of this theorem that will be used below in a solution of a particular gas-dynamic problem.

Theorem 1.2. *Let, for problem* (1.1), *the following conditions be satisfied*:

(1) *the functions a, b, c, d, e, f, g, h, p, q, r, s, and t are analytical in a vicinity of the point O;*

(2) $f_0 \neq n$, $s_0 \neq n$, $f_0 + r_0 E_0 k/(k - s_0) \neq n$, $n, k \in N$,

$$\begin{gathered} (1 - C_n)(1 - B_n) > A_n D_n > -B_n(1 - C_n), \quad 1 - B_n > 0, \quad 1 - C_n^* \geqslant 0, \\ C_0 \neq 1, \quad \gamma_0^2 > 4a_0, \quad (B_0 + a_0 - b_0)^2 > |a_0|. \end{gathered} \tag{1.14}$$

Then problem (1.1) *has a unique analytical solution.* Here $a_0 = A_0 D_0$, $b_0 = B_0 C_0$, and $\gamma_0 = 1 + a_0 - b_0$.

Remark. The condition $C_{n+1}^* \leqslant 1$ ensures satisfaction of the inequality $\Delta_n \neq 0$. The following generalization of this condition is valid: $\Delta_n \neq 0$, $n = 1, \ldots, n_0$; $C_n^* \leqslant 1$, $n = n_0, n_0 + 1, \ldots$ (the validity of the generalization follows from Lemma 1.4, which is formulated below).

Before proving Theorem 1.2, we formulate some auxiliary statements.

We consider a pair of sequences α_n and β_n ($n \in N$) calculated by the rule

$$\alpha_1 = 1, \quad \beta_1 = B_1, \quad \alpha_{n+1} = \alpha_n - C_{n+1}\beta_n,$$

$$\beta_{n+1} = A_{n+1}D_{n+1}\beta_n + B_{n+1}\alpha_{n+1} = B_{n+1}\alpha_n + (A_{n+1}D_{n+1} - B_{n+1}C_{n+1})\beta_n.$$

Lemma 1.1. *If $\Delta_n \neq 0$, $n \in N$,* then $\alpha_{n+1} = \prod_{k=1}^{n} \Delta_k$, $\quad n = 1, 2, \ldots$.

Lemma 1.2. *Let $(1 - C_n)(1 - B_n) > A_n D_n > -B_n(1 - C_n)$, $1 > B_n > 0$, $n \in N$. Then, $\alpha_n > \beta_n > 0$, $n \in N$.*

Lemma 1.3. *If $\Delta_n \neq 0$, $\Delta_n^* \neq 0$, $n \in N$, then $\alpha_{n+1}^* = \Delta_n^* \prod_{k=1}^{n-1} \Delta_k$, $n = 1, 2, \ldots$, where $\alpha_{n+1}^* = \alpha_n - C_{n+1}^*\beta_n$.*

Lemma 1.4. *If $\alpha_n > \beta_n > 0$, $C_n^* \leqslant 1$, $n \in N$, then $\alpha_{n+1}^* > 0$.*

Lemma 1.5. *If the sequence Δ_n converges, $4a_0 \leqslant \gamma_0^2$. Conversely, if $4a_0 < \gamma_0^2$, $a_0 \neq 0$, and $\Delta_n \neq 0$ ($n \in N$), the sequence Δ_n converges.*

The proofs of the lemmas are not given here.

Proof of Theorem 1.2. From the conditions of the theorem and Lemmas 1.1-1.4 it follows that conditions (1.3) and (1.4) are satisfied. In addition, according to Lemma 1.5, condition (1.5) is satisfied. We now prove that $\Delta_\infty^2 > |a_0|$. We use Lemma 1.2:

$$|\alpha_{n+1}| = |a_n - C_{n+1}\beta_n| > (1 - |C_{n+1}|)|\beta_n| = (1 - |C_{n+1}|)|B_n\alpha_{n-1}$$

$$+ (A_nD_n - B_nC_n)\beta_{n-1}| > \ldots > |B_1|(1 - |C_{n+1}|)\prod_{k=2}^{n}(|B_k + A_kD_k - B_kC_k|).$$

Hence,

$$\prod_{k=1}^{n}\Delta_k > B_1(1 - C_{n+1})\prod_{k=2}^{n}(B_k + A_kD_k - B_kC_k) > 0.$$

Since the last inequality is valid for any $n \in N$, we have $\Delta_\infty \geqslant |B_0| + |A_0D_0 - C_0B_0|$. Hence, $\Delta_\infty^2 > |a_0|$. Thus, all conditions of Theorem 1.1 are satisfied, and, hence, the statement of Theorem 1.2 is also valid.

In Theorem 1.2, we have formulated sufficient conditions of analytical solvability, which will be checked below in a solution of problems of gas dynamics. We now formulate a generalization of this theorem that will be directly used to solve a gas-dynamic problem.

Theorem 1.3. *The problem*

$$\begin{gathered} xw_x = (r_0u + s_0w + xt)\Big|_{y=0}, \\ xu_x = x(A_0u_y + B_0v_x) + uf_0 + E_0(r_0u + s_0w)\Big|_{y=0} + xp, \\ xv_y = x(C_0u_y + D_0v_x) + ug_0 + H_0(r_0u + s_0w)\Big|_{y=0} + xq, \\ z_y = r, \\ w(0) = 0, \quad u(0,y) = 0, \quad v(x,0) = 0, \quad z(x,0) = 0 \end{gathered} \tag{1.15}$$

has a unique locally analytical solution, if the following conditions are satisfied:

(1) *the functions t, p, q, and r depend on the independent variables x and y, the unknown functions u, v, w, and z, and their first derivatives;*

(2) *the functions t, p, q, and r are linear in the derivatives w_x, u_x, v_x, z_x, u_y, v_y, and z_y, and the coefficients of these derivatives vanish at the point O ($x = 0$, $y = 0$, $u = 0$, $v = 0$, $w = 0$, $z = 0$);*

(3) *the functions t, p, q, and r are analytical in a vicinity of the point O with respect to the corresponding variables;*

(4) *for the constants A_0, B_0, C_0, D_0, g_0, f_0, E_0, H_0, r_0, and s_0, conditions* (1.14) *are satisfied.*

Problem (1.15) differs from problem (1.10) only by the presence of an equation for z that does not contain a singularity. Therefore, the proof of Theorem 1.3 is generally similar to the proof of Theorem 1.2 and is not given here.

2. Gas-Dynamic Problem. We consider the system of gas-dynamic equations [15, 16] for an ideal polytropic gas with the equation of state $p = A^2(S)\rho^\gamma/\gamma$, where p is the pressure, S is the entropy [below, s denotes the function $A(S)$], ρ is the density, and $\gamma = \text{const} > 1$ is the polytropic exponent of the gas. We study cylindrically ($\nu = 1$) or spherically ($\nu = 2$) symmetric flows that depend on time t and the distance $r = (x_1^2 + \ldots + x_{\nu+1}^2)^{1/2}$ (x_1, x_2, and x_3 are spatial coordinates). As the sought functions $\mathbf{U} = \mathbf{U}(t, r)$ we take $\mathbf{U} = (\sigma, u, s)$, where $\sigma = \rho^{(\gamma-1)/2}$ and u is the speed of the gas. Then the speed of sound in the gas is defined by the relation $c = \sigma s$, and the system of gas-dynamics equations has the form

$$\begin{gathered} \sigma_t + u\sigma_r + \frac{\gamma - 1}{2}\sigma\Big(u_r + \nu\frac{u}{r}\Big) = 0, \\ u_t + \frac{2}{\gamma - 1}\sigma s^2\sigma_r + uu_r + \frac{2}{\gamma}\sigma^2 ss_r = 0, \qquad s_t + us_r = 0. \end{gathered} \tag{2.1}$$

We seek a piecewise-analytical solution of system (2.1) for the problem of smooth motion of a piston in the gas that generates a focused compression wave. For this problem, the flow configuration in the plane of the variables t and r is given in Fig. 1.

At the moment $t = t_0$ and at $0 \leqslant r \leqslant r_0$, the homogeneous gas is at rest. From the point A ($t = t_0$, $r = r_0$), an impenetrable piston begins to move smoothly in the gas (the curve AB is the trajectory of motion of the piston). A sonic characteristic (the straight line AO), separating the compression-wave region Ω_1 from the region of rest Ω_0 and having constant speed minus the speed of sound c_0 in the gas region Ω_0, begins to propagate in the homogeneous gas, which is at rest in the region Ω_0. The moment of focussing of the sonic characteristic is taken as $t = 0$. For an analytical law of motion of the piston, in a vicinity of the point A in the region Ω_1, there is a unique analytical solution of the piston problem [4, 5] that describes isentropic flow. Outside this vicinity in the region Ω_1, singularities of the gradient catastrophe type can arise.

If the law of motion of the piston is chosen in a special manner, the flow in the region Ω_1 is self-similar [1–3]: for the system of ordinary differential equations describing self-similar flows $\mathbf{U} = \mathbf{U}(\lambda)$, an integral curve that passes through appropriate singular points is constructed. Thus, in the region Ω_1, a compression wave is chosen. The curves AO, AB, and OC (the trajectory of motion of a reflected shock wave) are uniquely constructed from the compression wave. For these self-similar flows, the curve OC is a straight line, and in the region Ω_2 between the reflected shock wave and the axis $r = 0$, the compressed gas is homogeneous and is at rest again. In the region Ω_1, the gas parameters are constant on the straight lines $\lambda = \text{const}$, including $\sigma(0, r) = \text{const} > 0$ and $u(0, r) = \text{const} < 0$. It is clear that the self-similar flows $\mathbf{U} = \mathbf{U}(\lambda)$ cannot define profiles of the gas-dynamic parameters at the moment $t = 0$ in the more general case:

$$\sigma(0, r) = \sigma_0(r), \quad u(0, r) = u_0(r), \quad s(0, r) = s_0 = \text{const}.$$

If one assumes that for any $\sigma_0(r)$ and $u_0(r)$ in the region Ω_1 at $t \geqslant 0$, system (2.1) has a solution, then the curve OC is no longer a straight line, and for the gas flow in the region Ω_2, σ, u, and s are no longer constants.

The goal of the present study is as follows. First, using the initial conditions

$$\sigma(0, r) = \sigma_0(r), \quad \sigma_0(0) > 0, \quad u(0, r) = u_0(r), \quad u_0(0) < 0, \quad s(0, r) = s_0 = \text{const} > 0 \tag{2.2}$$

it is necessary to construct a solution of system (2.1) in the region Ω_1 and to relate it to the problem of focussing of the compression wave. The solution of problem (2.1) and (2.2), depending on the initial data, can be related to the problem of focussing of the rarefaction wave. However, a substantial gas-dynamic problem that can be a "prehistory" of such a rarefaction wave has not been found. Then, in the region Ω_2, it is necessary to construct another solution of system (2.1) for which $u(t, 0) = 0$. Simultaneously with construction of a solution in Ω_2, it is necessary to determine the unknown shock wave OC on which the flow constructed in Ω_1 and the flow sought in Ω_2 are connected by the Hugoniot relations [15, 16]. Since the flow in the region Ω_1 is isentropic, without loss of generality, we assume that $s_0 = 1$, and, hence, $\sigma = c$.

The procedure of constructing a solution of problem (2.1) and (2.2) in the region Ω_1 is described in detail in [10], and, therefore, we shall be brief in reasoning.

In system (2.1), we introduce new variables:

$$\zeta = t/r, \quad \chi = r. \tag{2.3}$$

The Jacobian of the replacement is $J = 1/r$. Replacement (2.3) is degenerate for $r = 0$.

Theorem 2.1. *If $\mathbf{U}_0(r)$ are analytical functions in a vicinity of the point $r = 0$, the Cauchy problem (2.1) and (2.2) written in the variables ζ and χ has a unique analytical solution in a vicinity of the point ($\zeta = 0, \chi = 0$):*

$$\mathbf{U}(\zeta, \chi) = \sum_{k=0}^{\infty} \mathbf{U}_{k1}(\chi) \frac{\zeta^k}{k!}, \quad \mathbf{U}_{01}(\chi) = \mathbf{U}_0(\chi). \tag{2.4}$$

Theorem 2.1 is a corollary of the Cauchy–Kowalewski theorem.

Series (2.4) are defined irrespective of the signs of the components of the vector $\mathbf{U}_0(\chi)$ in a certain complete vicinity of the point ($\zeta = 0, \chi = 0$). From the physical sense of problem (2.1) and (2.2), we must consider solutions for $\chi \geqslant 0$ for which $\sigma \geqslant 0$.

Along the axis $O\chi$, the region of existence of the solution "reaches" the point $\chi = \chi_*$, at which there

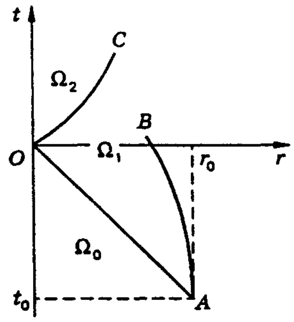

Fig. 1

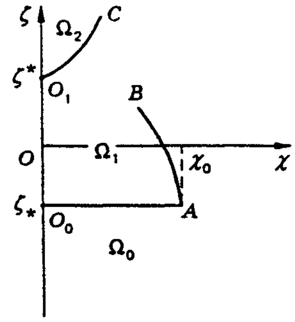

Fig. 2

is a singularity of the functions $\mathbf{U}_0(\chi)$ (it may be that $\chi_* = +\infty$). As $\chi \to \chi_*$, the radius of convergence of series (2.4) tends to zero as a certain positive power of the difference $\chi - \chi_*$ (or the fraction $1/\chi$, if $\chi_* = \infty$).

Along the axis $O\zeta$, the boundary points of the region of existence of an analytical solution are $\zeta_* < 0$ and $\zeta^* > 0$. In the case of focussing of the compression wave, the value of $\zeta = \zeta_*$ (the straight line AO_0 in Fig. 2) corresponds to the sonic characteristic AO in Fig. 1: $c(\zeta_*, \chi) = \text{const} > 0$, $u(\zeta_*, \chi) = 0$, and $c(\zeta_*, \chi) = -1/\zeta_*$. We note that in the case of focussing of the rarefaction wave, the value of $\zeta = \zeta_*$ corresponds to the free boundary (AO in Fig. 1): $c(\zeta_*, \chi) = 0$ and $u(\zeta_*, \chi) = \text{const} = 1/\zeta_*$.

In both cases, the value of $\zeta = \zeta^*$ is larger than the value $\zeta = \zeta_1$: $\zeta^* > \zeta_1 > 0$, where $1/\zeta_1$ is the speed of the reflected shock wave (the curve OC) in the case of self-similar flows. The value of $\zeta_1 = 1/D_0$ is uniquely determined from the equation

$$\frac{1}{\zeta_1} = \frac{3-\gamma}{4} u(\zeta_1, 0) + \left[\frac{(\gamma+1)^2}{16} u^2(\zeta_1, 0) + \sigma^2(\zeta_1, 0)\right]^{1/2}.$$

Transforming to dimensionless variables, it is easy to show that, without loss of generality, it is possible to assume that one of the values $\sigma_0(0)$ or $|u_0(0)|$ is equal to unity. Therefore, for given γ and ν, the value of $\mu = \sigma_0(0)/|u_0(0)|$ determines which of the quantities (σ or u) vanishes at the point ($\zeta = \zeta_*, \chi = 0$), i.e., which of the waves (compression or rarefaction) is focused. From the results of Sedov [1, p. 215] it follows that the statement below is valid.

Lemma 2.1. *For any $\mu > 0$ there are values of γ_1^* and γ_2^* such that*

— in the case of cylindrical symmetry, if $1 < \gamma < \gamma_1^*$, then $u(\zeta_*, 0) = 0$ and $\sigma(\zeta_*, 0) > 0$, i.e., the compression wave is focused, and if $\gamma_1^* < \gamma$ then $u(\zeta_*, 0) < 0$ and $c(\zeta_*, 0) = 0$, i.e., the rarefaction wave is focused;

— in the case of spherical symmetry, if $1 < \gamma < \gamma_2^*$, then $u(\zeta_*, 0) = 0$ and $\sigma(\zeta_*, 0) > 0$, i.e., the compression wave is focused, and if $\gamma_2^* < \gamma$, then $u(\zeta_*, 0) < 0$ and $\sigma(\zeta_*, 0) = 0$, i.e., the rarefaction wave is focused.

Numerical values of γ_1^* and γ_2^* are given below:

μ	0.1	0.25	0.5	1	2	4	10
γ_1^* ($\nu = 1$)	1.13	1.30	1.59	2.120	3.19	5.30	11.62
γ_2^* ($\nu = 2$)	1.10	1.24	1.45	1.835	2.65	4.25	9.04

Knowing the gas flow in the region Ω_1 as series (2.4), one can uniquely determine the trajectories of gas particle motion in the region Ω_1 by solving the appropriate Cauchy problems for ordinary differential equations. One of the particle trajectories constructed can be assumed to be the trajectory of motion of the impenetrable piston generating a wave compression in the region Ω_1.

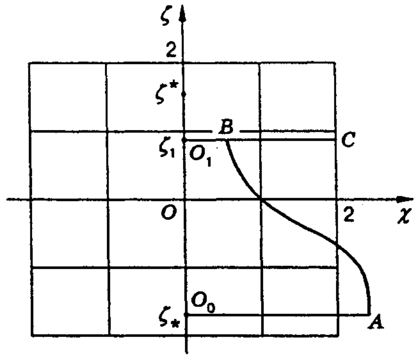

Fig. 3

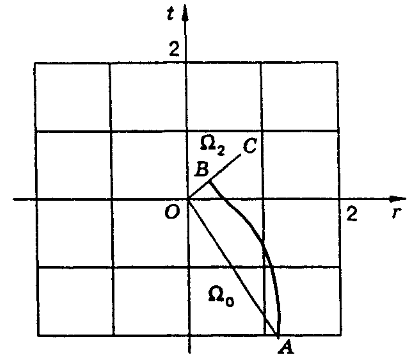

Fig. 4

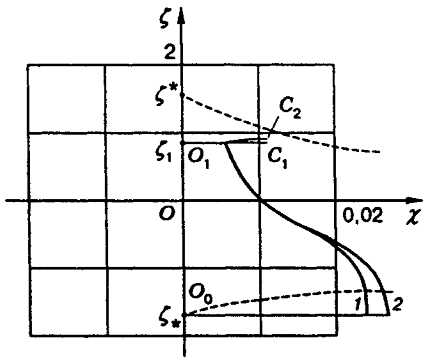

Fig. 5

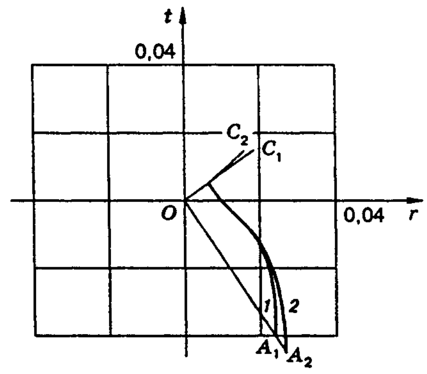

Fig. 6

If $u_0(r) = \text{const} < 0$ and $\sigma_0(r) = \text{const} > 0$, series (2.4) break at the first term, and the gas flow in the region Ω_1 is described by the self-similar Sedov solution. The differential equation for the particle trajectory in this case is integrated in quadratures. The particle trajectory that passes through the point $(\zeta = 0, \chi = \chi_0 = 1)$ is plotted in the plane of the variables ζ and χ in Fig. 3. In Fig. 4, the same trajectory is constructed in the plane of the variables t and r. In this case, the reflected shock wave (the straight lines O_1C and OC) moves at a constant speed, the gas in the regions Ω_0 and Ω_2 is homogeneous and is at rest, and $\sigma|_{\Omega_0} < \sigma|_{\Omega_2}$.

If $u_0(r)$ and $\sigma_0(r)$ are not constant, the coefficients of series (2.4) are different from zero for $n > 0$, and, hence, the flow in the region Ω_1 is not self-similar, and the reflected SW has a variable speed of motion.

Figures 5 and 6 show gas-particle trajectories in the region Ω_1 that pass through the point $\zeta = 0$, $\chi = \chi_0$, i.e., the point $t = 0$, $r = r_0 = \chi_0$ for the particular distribution of gas-dynamic parameters (curves 2), trajectories of particle motion in self-similar Sedov flow (curves 1), and trajectories of the reflected shock wave in the self-similar and non-self-similar cases (curves O_1C_1 and O_1C_2). The dashed curves in Fig. 5 illustrate the qualitative behavior of the region of convergence of series (2.4).

Let us proceed to constructing a solution of problem (2.1), (2.2) in the region Ω_2 and the law of motion of the reflected SW.

In system (2.1), we change both independent and dependent variables. Initially, using the formulas

$$r = \varphi(x), \quad t = y + x, \tag{2.5}$$

we replace r and t by the independent variables x and y. The Jacobian of mapping is $J = \varphi'(x)$. Here the function $r = \varphi(t)$ is as yet unknown and defines the trajectory of motion of the reflected SW. However, from the previous reasoning, we know the values of $\varphi(0) = 0$ and $\varphi'(0) = D(0) = 1/\zeta_1$. Hence, replacement (2.5) at the point ($t = 0$, $r = 0$) is nondegenerate, and, provided that the function $\varphi(x)$ is analytical, the replacement is also nondegenerate in a vicinity of the coordinate origin. In replacement (2.5), the axis $r = 0$ becomes the axis $x = 0$, and the SW line becomes the other coordinate axis $y = 0$.

The solution in the region Ω_2 is denoted by $\mathbf{U}$, and the solution in the region Ω_1 by $\mathbf{U}^1 = (u^1, c^1)$. Let us rewrite the Hugoniot conditions [16] on the shock wave (i.e., on the axis $y = 0$) in equivalent form for D, σ, and s in terms of $\mathbf{U}^1$ and u (this is possible by virtue of the "determinancy theorem" [16]):

$$
\begin{aligned}
D\Big|_{y=0} &= \left[\frac{3-\gamma}{4}u^1 + \frac{\gamma+1}{4}u + \sqrt{\frac{(\gamma+1)^2}{16}(u-u^1)^2 + (c^1)^2}\,\right]\Bigg|_{y=0},\\
\sigma\Big|_{y=0} &= \left\{c^1\left[\frac{(1/4)(\gamma+1)(u-u^1) + \sqrt{((\gamma+1)^2/16)(u-u^1)^2 + (c^1)^2}}{(1/4)(\gamma-3)(u-u^1) + \sqrt{((\gamma+1)^2/16)(u-u^1)^2 + (c^1)^2}}\right]^{(\gamma-1)/2}\right\}\Bigg|_{y=0},\\
s\Big|_{y=0} &= \left\{\left[\frac{1}{4}(\gamma-3)(u-u^1) + \sqrt{\frac{(\gamma+1)^2}{16}(u-u^1)^2 + (c^1)^2}\,\right]^{\gamma/2}\right.\\
&\quad\times\left[\frac{1}{4}(\gamma+1)(u-u^1) + \sqrt{\frac{(\gamma+1)^2}{16}(u-u^1)^2 + (c^1)^2}\,\right]^{(1-\gamma)/2}\\
&\quad\left.\times\left[\frac{1}{4}(3\gamma-1)(u-u^1) + \sqrt{\frac{(\gamma+1)^2}{16}(u-u^1)^2 + (c^1)^2}\,\right]^{1/2}\right\}\Bigg|_{y=0}.
\end{aligned}
\tag{2.6}
$$

Recall that $c = \sigma s$.

On the axis or at the center of symmetry we have the condition $u\big|_{x=0} = 0$. Therefore, the quantities $\sigma_{00} = \sigma\big|_{x=y=0}$ and $s_{00} = s\big|_{x=y=0}$ are uniquely determined from conditions (2.6), since $\mathbf{U}^1\big|_{\zeta=\zeta_1, \chi=0}$ are known from the preceding reasoning. Thus, $c_{00} = s_{00}\sigma_{00} > 0$.

We introduce the following designations:

$$
l = \frac{2}{\gamma-1}s_{00}\sigma + \frac{2}{\gamma}\sigma_{00}s, \quad M_0 = \frac{D_0}{\sigma_{00}\, s_{00}}.
$$

We note that under the Zemplén theorem [15], $0 < M_0 < 1$.

The functions u^1 and c^1 are determined along the unknown shock-wave front. Therefore,

$$
\mathbf{U}^1\Big|_{y=0} = \mathbf{U}^1(\zeta, \chi)\Big|_{y=0} = \mathbf{U}^1\Big(\frac{x+y}{x\psi(x)}, x\psi\Big)\Big|_{y=0} = \mathbf{U}^1\Big(\frac{1}{\psi(x)}, x\psi(x)\Big),
$$

where $\psi(x)$ is determined from the relation $\varphi(x) = x\psi(x)$. We designate the right parts of the Hugoniot conditions (2.6) by $D^*\big|_{y=0}$, $\sigma^*\big|_{y=0}$, and $s^*\big|_{y=0}$, respectively. By virtue of the aforesaid, we can write these functions as

$$
D^*|_{y=0} = \alpha u + \varepsilon\psi + q_0, \quad \sigma^*|_{y=0} = \alpha_1 u + \varepsilon_1\psi + q_1, \quad s^*|_{y=0} = \alpha_2 u + \varepsilon_2\psi + q_2.
$$

Here $(\partial q_i/\partial u)\big|_{x=y=0} = (\partial q_i/\partial\psi)\big|_{x=y=0} = 0$, $i = 0, 1, 2$.

The expressions for α, α_1, α_2, ε, ε_1, and ε_2 are rather cumbersome. Some of them, necessary in what follows, will be given below.

We introduce new unknown functions by the formulas

$$
u' = u, \quad v = -\frac{1}{\beta+1}(l - \beta u - e_0\psi - q_3), \quad w = \psi - D_0, \quad z = s - s^*\Big|_{y=0},
\tag{2.7}
$$

i.e., instead of u, σ, s, and ψ, we seek u', v, w, and z. Here

$$
\beta = \frac{2}{\gamma-1}s_{00}\alpha_1 + \frac{2}{\gamma}\sigma_{00}\alpha_2; \quad e_0 = \frac{2}{\gamma-1}s_{00}\varepsilon_1 + \frac{2}{\gamma}\sigma_{00}\varepsilon_2; \quad q_3 = \frac{2}{\gamma-1}s_{00}q_1 + \frac{2}{\gamma}\sigma_{00}q_2.
$$

Replacement (2.7) makes it possible to proceed to the Cauchy problem with zero initial conditions on different surfaces for a quasilinear system with singularities.

Solving the system resulting from replacements (2.5) and (2.7) for u'_x, v'_y, and w_x (for convenience, the prime is omitted below) we obtain

$$\begin{gathered} w_x = \left[\alpha \frac{u}{x} + (\varepsilon - 1)\frac{w}{x} + Y_1\right]\Big|_{y=0}, \\ u_x = \frac{1 - M_0}{1 + \beta M_0} u_y + \frac{M_0(1+\beta)}{1+\beta M_0} v_x - \frac{\nu}{1+\beta M_0}\frac{u}{x} - \frac{M_0 e_0}{1+\beta M_0} w_x + Y_2, \\ v_y = \frac{M_0(\beta - 1)}{1+\beta M_0} u_y + \frac{1}{1+\beta M_0} v_x + \frac{\nu\beta}{(1+\beta M_0)(1+\beta)}\frac{u}{x} - \frac{e_0}{(1+\beta M_0)(1+\beta)} w_x + Y_3, \\ z_y = Y_4. \end{gathered} \tag{2.8}$$

Here Y_i $(i = 1, \dots, 4)$ are specified functions for which conditions (1)–(3) of Theorem 1.3 are satisfied. The expressions for these functions are cumbersome and are not given here.

The condition on the symmetry axis for the gas speed $u = 0$ and the Hugoniot conditions on the SW in the new variables are written as

$$w(0) = 0, \quad u(0, y) = 0, \quad v(x, 0) = 0, \quad z(x, 0) = 0. \tag{2.9}$$

Thus for system (2.8), we obtain a Cauchy problem with initial data (2.9) on different surfaces: the initial values for the unknowns $w(x)$ and $u(x,y)$ are specified on the coordinate axis $x = 0$, and for the other two functions $v(x,y)$ and $z(x,y)$, they are specified on the other coordinate axis $y = 0$. Recall that in the space of the physical variables t and r, the straight line $r = 0$ corresponds to the line $x = 0$, the initial conditions on this straight line correspond to the equalities $D(t)|_{t=0} = 1/\zeta_1$ and $u|_{r=0} = 0$, and the trajectory of the unknown SW corresponds to the line $y = 0$. Thus, two of the three Hugoniot conditions on the SW become the initial data for $v(x,y)$ and $z(x,y)$ specified on the straight line $y = 0$, and the third Hugoniot condition becomes the first equation of system (2.8). Problem (2.8) and (2.9) describes flows in the region Ω_2 that exactly satisfy the Hugoniot conditions.

Theorem 2.2. *Problem* (2.8), (2.9) *has a unique analytical solution for* $\gamma \geqslant \gamma_0$, *where*

$$\gamma_0 = 1{,}117\,749\ldots \quad \text{for } \nu = 2, \qquad \gamma_0 = 1{,}051\,854\ldots \quad \text{for } \nu = 1. \tag{2.10}$$

For $\gamma \geqslant \gamma_0$, *problem* (2.1), (2.2) *has a unique analytical solution in the region* Ω_2 *that also determines the trajectory of the reflected SW on which the Hugoniot conditions are satisfied. For the solution in* Ω_2, *the symmetry condition is also satisfied.*

Theorem 2.2 is proved by means of Theorem 1.3. We verify that the conditions of this theorem are satisfied for problem (2.8), (2.9).

Conditions (1)–(3) of Theorem 1.3 are satisfied because the functions D^*, σ^*, and s^* are analytical in a vicinity of the point $(x = 0, y = 0, u = 0, v = 0, w = 0, z = 0)$. We verify satisfaction of conditions (1.14).

We write the following necessary constants:

$$\begin{gathered} A_0 = \frac{1 - M_0^2}{1 + \beta M_0}, \quad B_0 = \frac{M_0(1+\beta)}{1+\beta M_0}, \quad C_0 = \frac{M_0(\beta - 1)}{1 + \beta M_0}, \quad D_0 = \frac{1}{1+\beta M_0}, \\ g_0 = \frac{\nu\beta}{(1+\beta M_0)(1+\beta)}, \quad f_0 = -\frac{\nu}{1+\beta M_0}, \quad E_0 = -\frac{M_0 e_0}{1+\beta M_0}, \quad H_0 = -\frac{e_0}{(1+\beta)(1+\beta M_0)}, \\ r_0 = \alpha, \quad s_0 = \varepsilon - 1. \end{gathered} \tag{2.11}$$

Conditions (1.14), except for the inequality $1 \geqslant C^*_{n+1}$, were verified in [10].

The proof of the validity of the inequality $1 \geqslant C^*_{n+1}$, i.e., the estimation of the terms of the sequence

C_{n+1}^* ($n \in N$) taking into account (2.11)

$$C_n^* = \frac{M_0(\beta - 1)}{1 + \beta M_0} + \frac{1 - M_0^2}{(1 + \beta M_0)(\beta + 1)}\Big(\nu\beta - \frac{e_\alpha n}{n + 1 - \varepsilon}\Big)\frac{1}{n(1 + \beta M_0) + \nu + M_0 e_\alpha n/(n + 1 - \varepsilon)}$$

($e_\alpha = \alpha e_0$), is rather labor-consuming and is given here only briefly.

We write explicit formulas for the constants e_0, β, α, ε, and M_0 using the solution in the region Ω_1 and the Hugoniot conditions (2.6):

$$e_0 = -\frac{\nu}{2}\frac{\delta - (\gamma - 1)/2}{(\delta - (\gamma + 1)/4)[(\delta - 1)(\delta + (\gamma - 1)/2)]^{1/2}} < 0,$$

$$\beta = \Big(\frac{\delta - 1}{\delta + (\gamma - 1)/2}\Big)^{1/2}\frac{\delta}{\delta - (\gamma + 1)/4},$$

$$\alpha = \frac{(\gamma + 1)}{4}\frac{\delta}{(\delta - (\gamma + 1)/4)} > 0, \quad \varepsilon = -\frac{\nu}{2}\frac{(\delta - (\gamma + 1)/2)}{(\delta - (\gamma + 1)/4)} < 0, \tag{2.12}$$

$$M_0 = \Big(\frac{\delta - 1}{\delta + (\gamma - 1)/2}\Big)^{1/2}, \qquad \delta = \frac{\gamma + 1}{4} + \frac{1}{u - u^1}\sqrt{\frac{(\gamma + 1)^2}{16}(u - u^1)^2 + (c^1)^2}.$$

Using relations (2.12), is possible to prove the validity of the inequality $C_{n+1}^* \leqslant 1$ if γ satisfies (2.10).

Thus, Theorem 2.2 is proved. Problem (2.8), (2.9) has a unique analytical solution if $\gamma \geqslant \gamma_0$, where γ_0 is determined from (2.10).

It can be suggested that problem (2.8), (2.9) has a unique analytical solution if $1 < \gamma < \gamma_0$ for any analytical functions $u_0(x)$ and $\sigma_0(x)$. However, a new theorem which is different from Theorem 1.3 is required to prove this.

The author is grateful to S. P. Bautin for useful discussions.

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